5. Circuit solving techniques

Linear Systems and Circuit Theory Applications

Óscar Barquero

Universidad
Rey Juan Carlos

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Contents

Introduction

- Ohm and Kirchhoff Laws together with the circuits' simplification techniques (elements association, sources transformation, voltaje and current dividers, etc.), seen in the previous block, allow us to analyze simple resistive circuits.

- As the circuits become more complicated, their resolution by means of the previous methods is not very effective, so in this topic, two analytical methods allowing the analysis of more complex structures will be presented: nodal analysis and mesh analysis.

- In addition, two new powerful simplication techniques will be presented: Thévenin and Norton equivalents.

- Design rules for máximum power transfer from a source to a resistive load will be given.

Introducción. Number of simultaneous equations

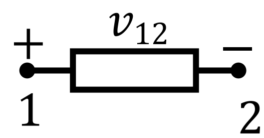
- To solve a circuit with b unknown current values, b independent equations are required (b is the number of branches in which the current is unknown).
- For a circuit having n nodes, n-1 independent equations are obtained when applying the Kirchhoff law of the currents.
- The b- (n-1) remaining equations will be obtained by applying the Kirchhoff law of the voltages in the loops or meshes of the circuit.
- A systematic method is established to determine the number of necessary equations from the number of nodes, meshes and branches in which the current is unknown.
  - Kirchhoff law of the currents is applied to n-1 nodes and Kirchhoff law of the tensions to b-(n-1) loops or meshs.
  - Previous statement can be particularized for essential nodes n_e and essential branches b_e, and, since the number of essential nodes and branches is always less than the number of regular nodes and branches, a smaller equation set is obtained when working with essential nodes and branches:
    - We apply Kirchhoff Law of currentes to n_e-1 essential nodes and Kirchhoff Law of voltaje to b_e-(n_e-1) loops or meshes.

Introducción. Number of simultaneous equations

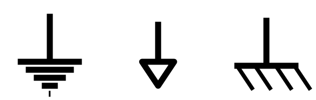
- By means of a previous study of the circuit and a careful selection of the nodes and the meshes, it is possible to further reduce the number of equations necessary to solve the circuit.
- It is possible to go a step further and, from the definition of new variables, reduce the number of necessary equations to only n-1 or only b- (n-1) equations.
- These new variables are called node voltages and mesh currents.
- Nodal analysis will allow us to solve the circuit in terms of n-1 equations.
- Mesh analysis will allow us to solve the circuit in terms of b- (n-1) equations.

Introduction. Node voltage

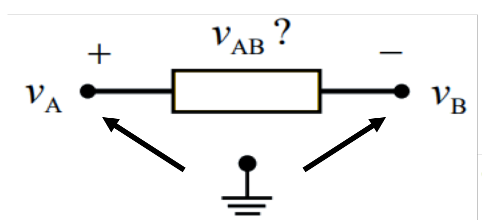
- So far we have worked with voltage or potential in terms of "potential difference between two nodes" which, generally, corresponds to the terminals of a circuit element.



- The node voltage is defined as the potential drop of a node with respect to a node called reference node, of zero potential.
  - The reference node is usually also referred to as a ground or ground node.
  - Its choice is arbitrary, however the node to which more branches arrive is usually chosen.
  - The reference node is usually identified with symbols:



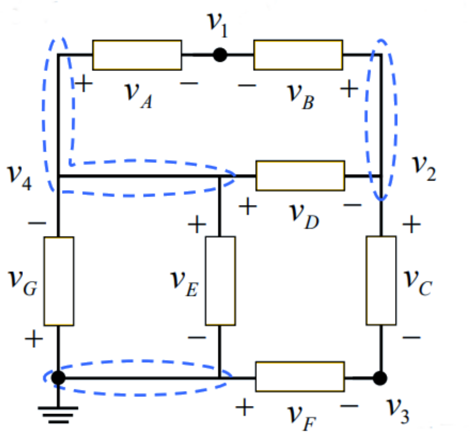
  - Once the node voltages are known in all the nodes of a circuit, it is possible to easily obtain the voltage drop at each element.



$$v_{AB} = v_A - v_B$$

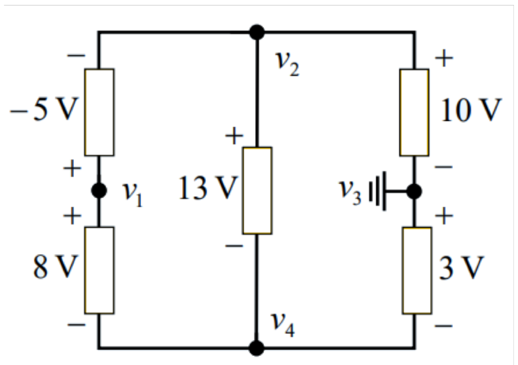
Introduction. Node-voltage

- **Example 1:** obtain voltaje drops and increments at each element when v_1=10V, v_2=2V, v_3=-4V y V_4=5V.



- $v_a=v_4-v_1=5-10=-5V$
- $v_b=v_2-v_1=2-10=-8V$
- $v_c=v_2-v_3=2-(-4)=6V$
- $v_d=v_4-v_2=5-2=3V$
- $v_e=v_4-0=5V$
- $v_f=0-v_3=4V$
- $v_{g=0\text{-}}v_4=-5V$

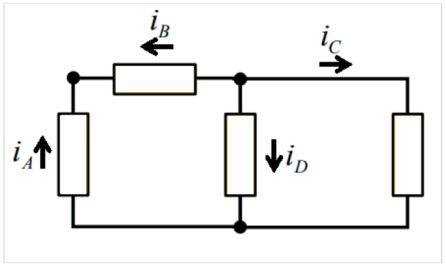
- **Example 2:** obtain node voltajes in the following circuit when node v3 is selected as reference node:



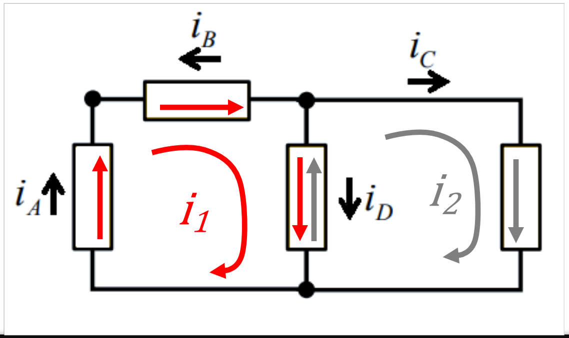
- $v_3=0V$
- $v_2-v_3=10V \rightarrow v2=10V$
- $v_4-v_3=-3V \rightarrow v4=-3V$
- $v_1-v_4=8V \rightarrow v1= 5V$

Introduction. Mesh current

- In the previous block, we worked with branch currents, that is, currents flowing between two nodes, normally associated with a specific element:



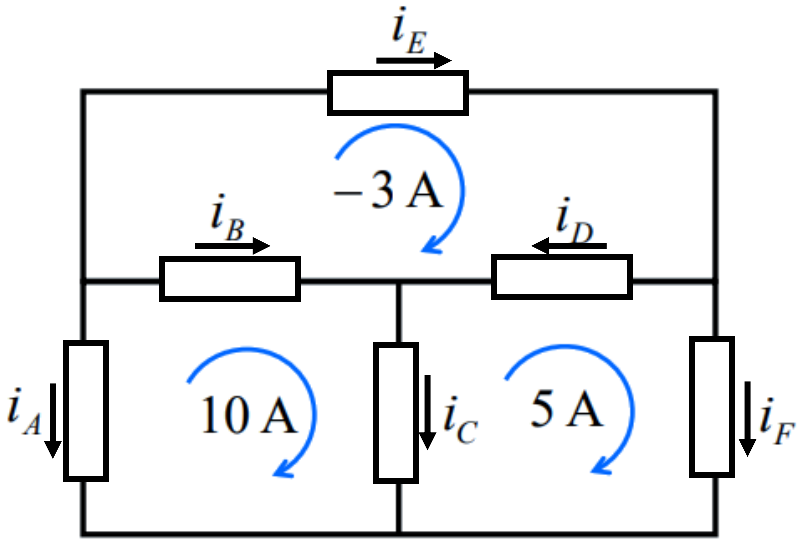
- Alternatively we can introduce the concept of mesh current as the closed current that runs through a certain mesh. The assignment of the direction of rotation of the current in each mesh is arbitrary.
  - For a given circuit, the relationship between branch currents and mesh currents can be determined by inspection:



- $i_A=i_1$
- $i_B=-i_1$
- $i_C=i_2$
- $i_D=i_1-i_2$

Introduction. Mesh current

- Example: Obtain the current at each branch from the known mesh currents.



- i_A= -10 A
- i_B=-10-3=-13 A
- i_C=-10+5=-5 A
- i_D=-3-5=-8 A
- i_E=-3 A
- i_F=-5 A

Nodal analysis and Mesh analysis

Nodal analysis

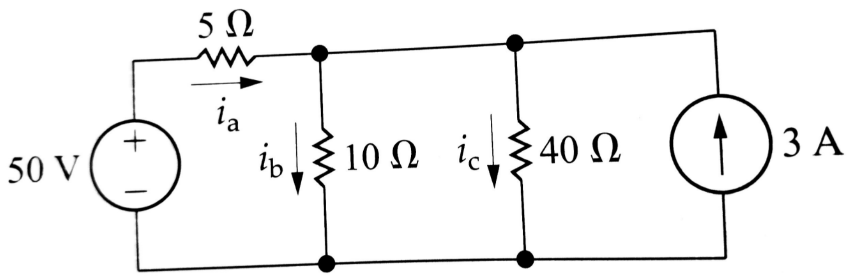
- Goal: to obtain voltaje of each node of the circuit when the reference node-voltaje is known.
  - Once all the node voltages are known, all the branch currents and powers of each circuit element can be calculated.

- Methodology:

1) Locate the n_e essential circuit nodes (n_e - 1 equations needed).
2) Choose one of the essential nodes as reference node (null voltage).
3) Define voltage nodes.
4) Apply the Kirchhoff Law of the currents on the essential nodes (except the reference node) expressing the currents entering and leaving each node as a function of the node voltaje. Define the direction of the currents previously.
5) Solve the n_e -1 equation set (n_e -1 unknowns).

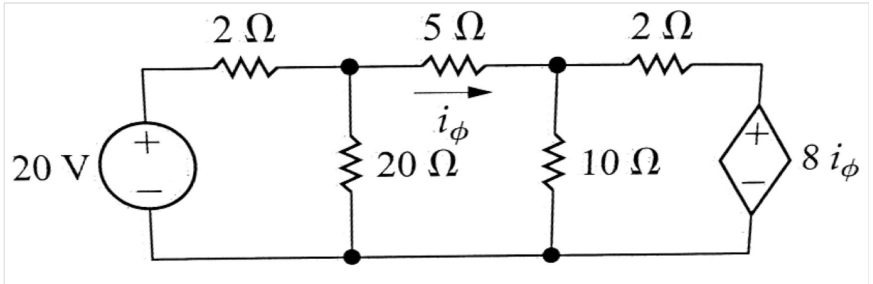
Nodal analysis

- Example: obtain the branch currents i_a, i_b e i_c by means of nodal analysis. Obtain power at each of the sources.

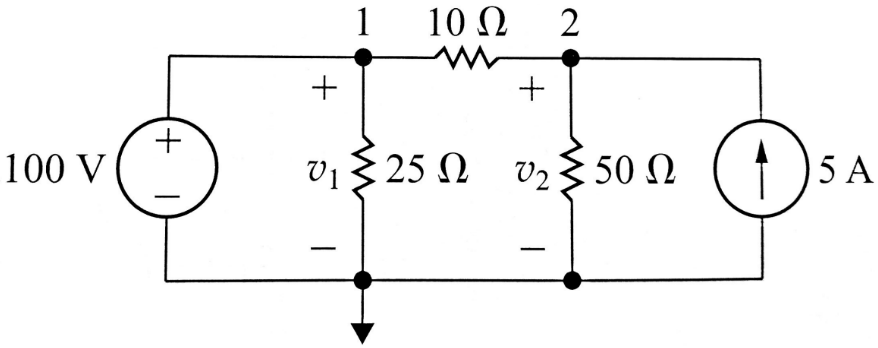


Nodal analysis. Particular cases

- If the circuit contains **dependent sources**, restrictions imposed by the control equations of these sources must be added to equation set as new equations.



- If the **only element between two essential nodes is a voltage source**, the number of equations is reduced by one since the voltage source will impose the value of the node voltage between those essential nodes.



Mesh analysis

- Goal: to obtain each meash current.
  - Once all mesh currents are known, the branch currents and the voltages and / or powers of each circuit element can be calculated.

- Methodology:

1) Locate the M cicuit meses (M equations will be needed).

2) Assign an arbitrary sens of current circulation and name all the mesh currents (I_1, I_2, …, I_M).

3) Apply Kirchhoff law of the voltage to each mesh.

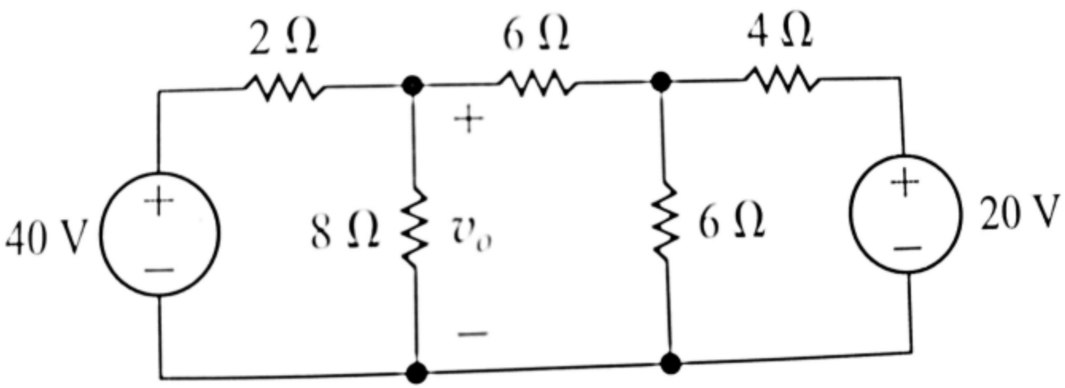
   When going through each mesh, we consider that we go through all the resistances in the direction of the voltaje drop.

4) Solve the M equation set (M unknowns).

WARNING: This method can only be applied to planar circuits, that is, circuits that can be drawn in a plane without crossing any of its branches.

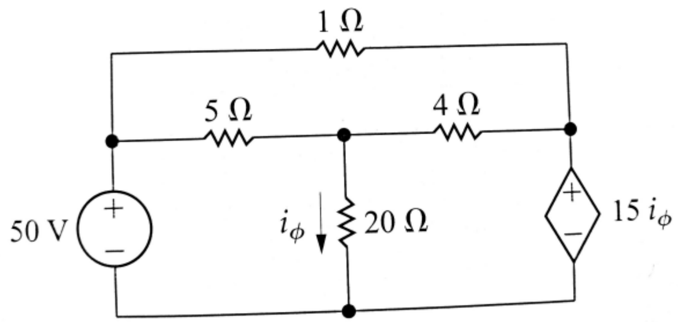
Mesh analysis

- Example: Solve the circuit by means of mesh analysis. Obtain the voltaje drop at the 8 Ohm resistor and the power at each voltaje source.

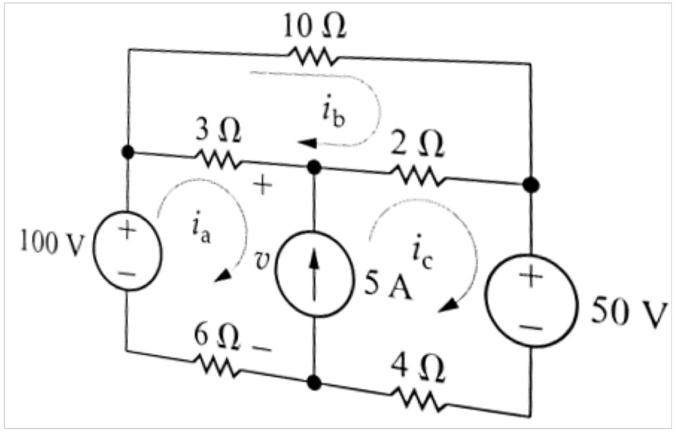


Mesh analysis. Particular cases

- If the circuit has **dependent sources**, restrictions imposed by the control equations of these sources must be added to the equations set.



- When **a branch contains only a current source**, the application of the method requires certain additional manipulations.
  - The number of equations can be reduced by one.
  - However, a new variable is needed in order to model the voltaje drop in the current source.



Solving technique selection

- When is it better to use mesh analysis and when is it better to use node voltaje analysis?

  - Does any of the methods provide a smaller number of equations

    - Inspect the circuit and calculate the number of meshes and essential nodes.

  - Is it posible to apply some of the simplifications previously seen?

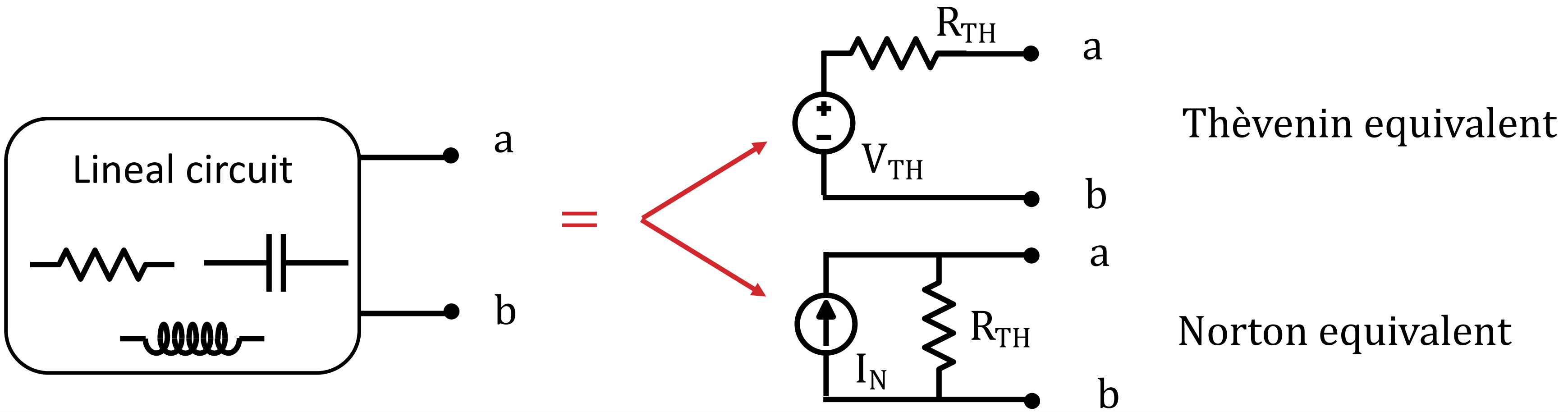
    - If the circuit has a voltage source as the only element between two essential nodes, it will be possible to reduce the number of equations of the equation set of the node voltage analysis in one.

    - If the circuit has only a current source in a branch, it will be possible to reduce the number of equations of the equation set of the mesh analysis in one.

Thèvenin and Norton Equivalents

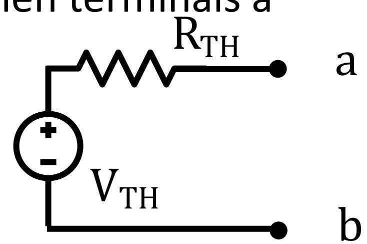
Thèvenin and Norton equivalents

- Sometimes, we are interested in knowing what happens between a pair of terminals of a circuit, without knowing what happens in the rest of the circuit:
    - Example: mobile phone charger plugged into the power grid. What interests us is the voltage and current in the terminals of the charger to which we will connect the mobile, not the circuit itself.
- Thévenin and Norton equivalents are circuit simplification techniques that model the behavior of two determined terminals and that can be applied to any linear circuit.
- Any linear network, in a given pair of nodes, can be replaced by an equivalent circuit:



Thèvenin and Norton equivalents

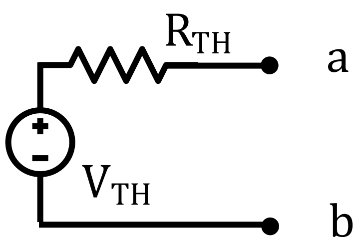
- To represent the original circuit by its Thévenin equivalent it is necessary to obtain V_{TH} y R_{TH}.
- If the load resistance (resistance that is connected to terminals a and b) is infinitely large, we are in open circuit condition. The open circuit voltage between terminals a and b is, by definition, V_{TH}.
  - This voltage must be the same as the voltage between the terminals a b of the original circuit when these are left in open circuit.
  - To calculate V_{TH} we need to solve for the open circuit voltage in the original circuit.
- If we shortcircuit terminals a-b, the shortcircuit current i_{sc} from a to b will be $i_{sc} = \frac{V_{TH}}{R_{TH}}$.
  - This current must be the same as the current in the original circuit when terminals a and b are short-circuited.
  - R_{TH} is the quotien between V_{TH} and the short-circuit current between terminals a and b of the original circuit.



Thèvenin equivalent

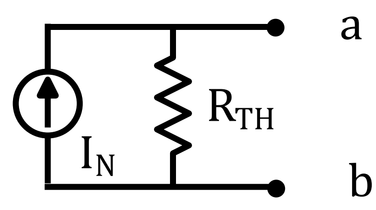
Thèvenin and Norton equivalents

- Norton equivalent can be obtained from the Thèvenin equivalent by simply applying the transformation techniques seen in the previous block of the course.



Thèvenin equivalent

$$I_N = \frac{V_{TH}}{R_{TH}}$$

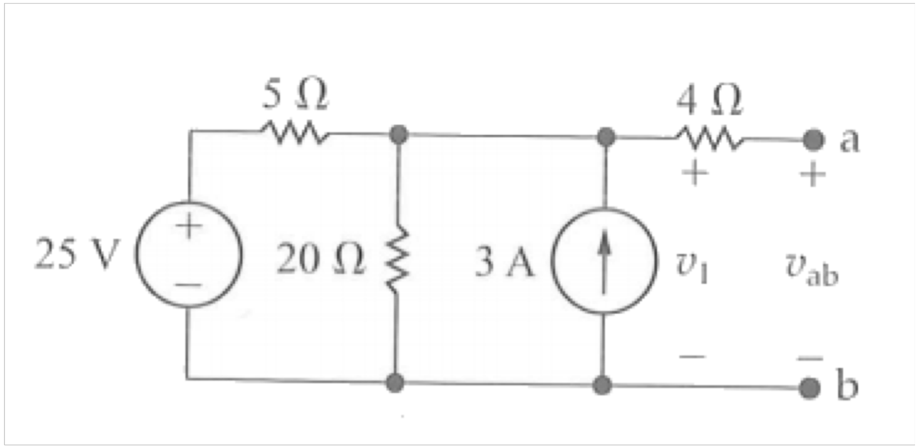


Norton equivalent

- Thèvenin and Norton equivalents are not unique and depend on the selected pair of terminals.

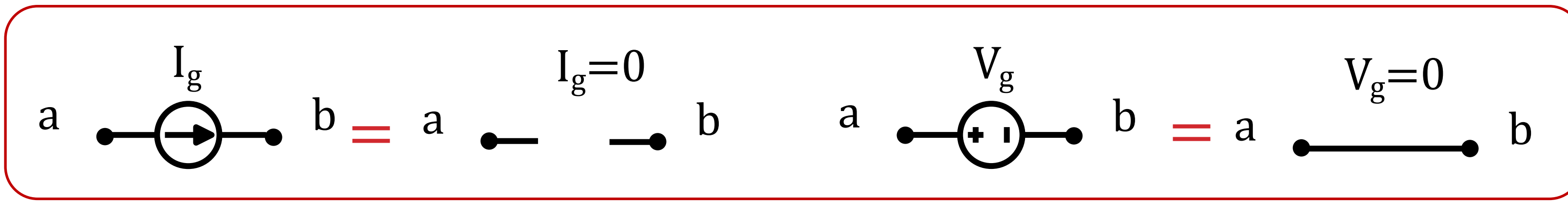
Thèvenin and Norton equivalents

- Example: Obtain the Thèvenin equivalent between a and b of the following circuit:

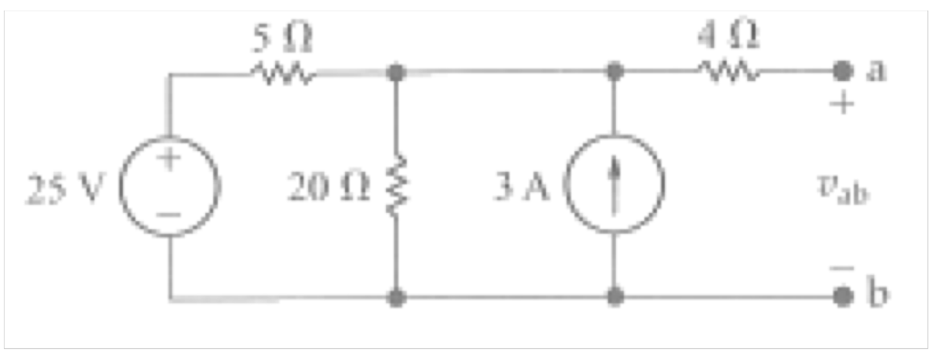


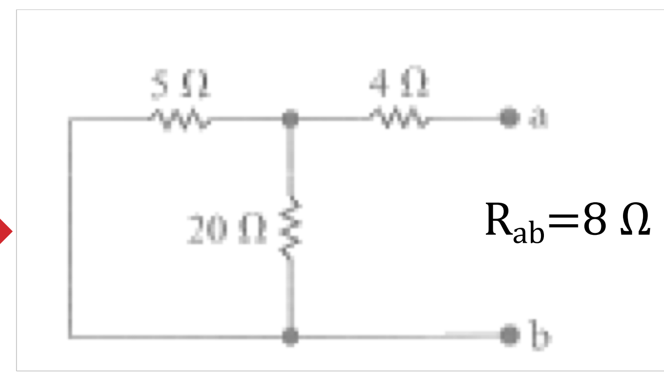
Thèvenin and Norton equivalents. Simplifications

- If there are only independent sources in the circuit, calculation of the Thèvenin resistance is simplified.
  - To obtain R_{TH} all independent sources are deactivated and the circuit resistance between the selected pair of terminals is calculated.
  - A voltage source is deactivated by replacing it with a short circuit.
  - A current source is deactivated by replacing it with an open circuit.



- Example:

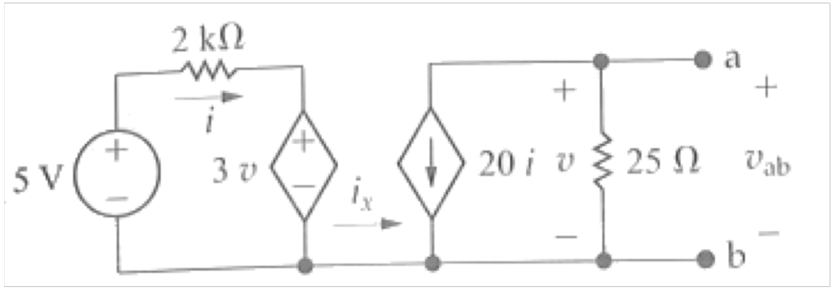


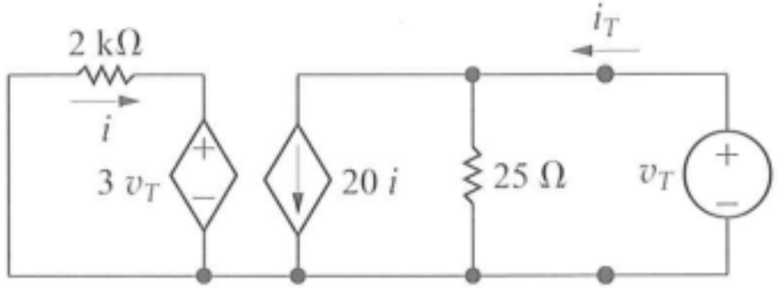


Thèvenin and Norton equivalents. Simplifications

- If there are dependent sources in the circuit:
  - All independent sources are deactivated,
  - The circuit is excited with a test source supplying $\mathrm{V_t}$ and $\mathrm{I_t}$,
  - Thèvenin resistor is computed as $\mathrm{R}_{TH} = \frac{V_T}{R_T}$.

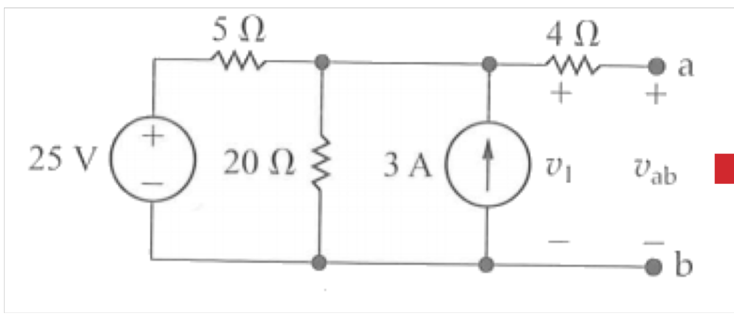
- Example: In this case we use a voltage source that allows us to know the voltage drop in the voltage-dependent source and obtain the control current i from the current source.

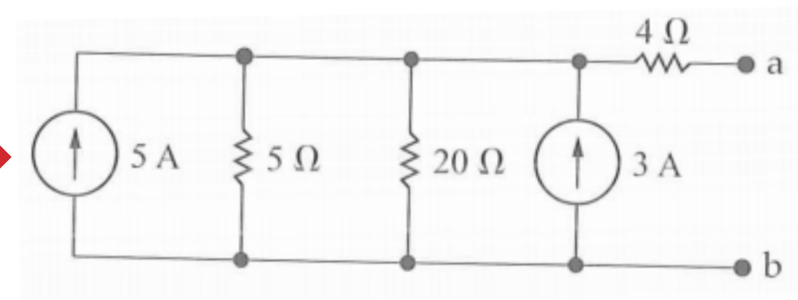


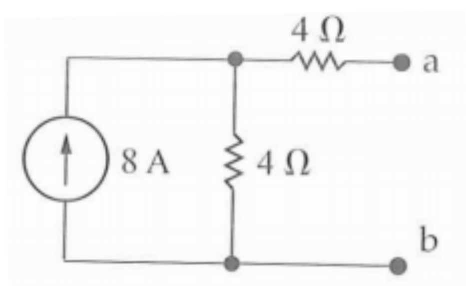


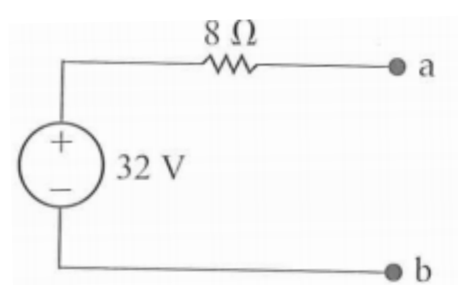
Thèvenin and Norton equivalents. Sources transformation

- Sometimes it is possible to obtain the equivalent through the source transformation technique. This technique is especially useful in the case that the circuit only contains independent sources.
- Example: try to obtain the Thévenin and Norton equivalents of the previous example:

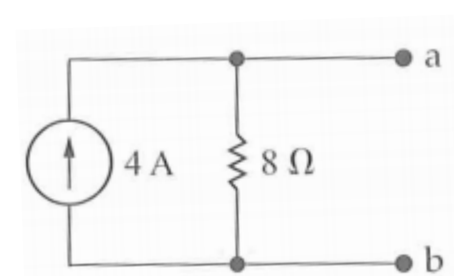








EQ. THÈVENIN

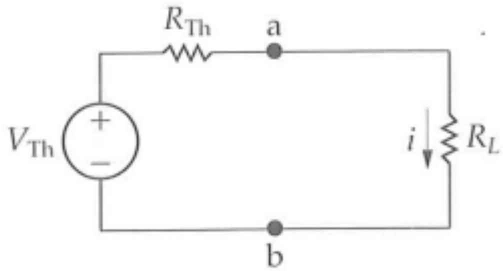


EQ. NORTON

Maximum power transfer

Maximum power transfer

- One of the main goals in circuit design is efficient power delivery from a source to a load.
  - A load is any passive element connected to the circuit terminals.
- The starting point is the Thèvenin equivalent of a resistive network containing dependent and independent sources to which a load R_L (L = LOAD) is connected and we calculate the expression of the power in the load:

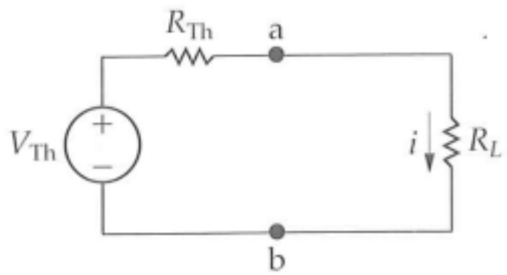


$$p = i^2 R_L = \left(\frac{V_{Th}}{R_{Th} + R_L}\right)^2 R_L$$

- For a given circuit V_{TH} y R_{TH} values are fixed, the only value that can be optimized to achieve maximum power transfer is the value of the load resistance.

Maximum power transfer

- To compute the load resistance with which the máximum power transfer is obtained, we begin by calculating the first derivative of the power with respect to the load:



$$p = i^2 R_L = \left(\frac{V_{Th}}{R_{Th} + R_L}\right)^2 R_L$$

$$\frac{dp}{dR_L} = V_{Th}^2 \left[\frac{(R_{Th} + R_L)^2 - R_L \cdot 2(R_{Th} + R_L)}{(R_{Th} + R_L)^4}\right]$$

- P will be maximized when said derivative is null, which occurs when:

$$(R_{Th} + R_L)^2 = 2R_L(R_{Th} + R_L).$$

$$R_L = R_{Th}.$$

- Maximum power transfer occurs when the load resistance is equal to the Thèvenin resistance of the circuit. In that case, the power delivered to the load will be:

$$p_{máx} = \frac{V_{Th}^2 R_L}{(2R_L)^2} = \frac{V_{Th}^2}{4R_L}$$

Maximum power transfer

- Example: Calculate the value of the load resistance that results in a maximum transfer of power to the load and calculate that power. What percentage of the power supplied by the voltage source dissipates in the load?